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Binding energy of semirelativistic N-boson systems

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Abstract

General analytic energy bounds are derived for N-boson systems governed by semirelativistic Hamiltonians of the form

$$H = \sum_{i=1}^{N} \left(\mathbf{p}_i^2 + m^2\right)^{1/2} + \sum_{1=i<j}^{N} V(r_{ij}),$$

where $V(r)$ is a static attractive pair potential. A translation-invariant model Hamiltonian H_c is constructed. We conjecture that $\langle H\rangle \geqslant \langle H_c\rangle$ generally, and we prove this for $N = 3$, and for $N = 4$ when $m = 0$. The conjecture is also valid generally for the harmonic oscillator and in the nonrelativistic large-m limit. This formulation allows reductions to scaled 3- or 4-body problems, whose spectral bottoms provide energy lower bounds. The example of the ultrarelativistic linear potential is studied in detail and explicit upper- and lower-bound formulae are derived and compared with earlier bounds.

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1. Introduction

One-body Hamiltonians H composed of the relativistic expression $\sqrt{\mathbf{p}^2 + m^2}$ for the kinetic energy of particles of mass m and momentum $\mathbf{p}$ and of a coordinate-dependent static interaction potential $V(\mathbf{r})$, defined as operator sum

$$H = \sqrt{\mathbf{p}^2 + m^2} + V(\mathbf{r}),$$

provide a simple but very efficient tool for the description of relativistically moving particles [1–3]. They have been used, for instance, for the description of hadrons as bound states of quarks [4]. One of the advantages of this kind of semirelativistic treatment is that its generalization to the many-body problem is straightforward to formulate [5]. A semirelativistic

Hamiltonian for a system of N identical particles interacting by pair potentials $V(r_{ij})$ is given by

$$H = \sum_{i=1}^{N} \sqrt{p_i^2 + m^2} + \sum_{1=i<j}^{N} V(r_{ij}). \tag{1.1}$$

We use the notational simplification $p \equiv \|\mathbf{p}\|, r \equiv \|\mathbf{r}\|$, or $r_{ij} \equiv \|\mathbf{r}_i - \mathbf{r}_j\|$, whenever no ambiguity is introduced by so doing. Many approaches to such many-body problems for identical particles employ the very powerful constraint of permutation symmetry to generate their reduction to a 2-body problem with a Hamiltonian $\mathcal{H}$ whose spectrum is used to approximate the many-body energy eigenvalues or to generate a lower energy bound. This reduction may be effected in various ways, which leads to the problem of finding the most effective reduced problem, the one which would provide the *highest* lower bound. In one analysis [6] involving pseudo-fermions (where the necessary permutation antisymmetry is carried entirely by the spatial part of the wavefunction), an optimization is considered over a real parameter which characterizes the degree of orthogonality of the matrix B that defines the relative coordinates. For boson systems, an orthogonal B is best possible; one such choice is the Jacobi coordinate system that we shall use in section 2.

For the boson problem, perhaps the most immediate reduction is what we have called the simple or $N/2$ bound based on the equality $\langle H\rangle = \langle H_2\rangle$, where

$$H_2 = \frac{N}{2}\left[\sqrt{p_1^2 + m^2} + \sqrt{p_2^2 + m^2} + (N-1)V(r_{12})\right]. \tag{1.2}$$

The $N/2$ bound is then the bottom E_2 of the spectrum of the scaled 2-body Hamiltonian H_2. We have explicitly for this bound

$$E \geqslant E_{N/2}^L = N \inf_{\psi}\left(\psi, \left[(p^2 + m^2)^{\frac{1}{2}} + \frac{N-1}{2}V(r)\right]\psi\right). \tag{1.3}$$

If this reasoning is applied to the Schrödinger harmonic-oscillator problem, one finds for large-N that $E_{N/2}^L \to E/\sqrt{2}$, whereas a reduction based on Jacobi coordinates [7] yields $E_L = E$. We note parenthetically that the $N/2$ bound is equivalent to using a non-orthogonal coordinate system consisting of a centre-of-mass coordinate and $N-1$ pair distances [8]. Similarly, one obtains dramatic improvement over the $N/2$ lower bound if Jacobi coordinates are used for the Salpeter harmonic-oscillator problem [9]. We have obtained improved lower bounds for potentials which are convex transformations $V(r) = g(r^2)$ of the oscillator [10], and also, by very special reasoning, for the gravitational potential [11], $V(r) = -v/r, v > 0$. In the present paper, we look for good lower bounds that are valid for general attractive potentials, for example, of the form $V(r) = -v/r + br, v \geqslant 0, b > 0$.

Since the spectrum of the semirelativistic many-body Hamiltonian H can be characterized variationally, it is straightforward to find upper energy bounds with the aid of a suitable trial function. The principal difficulty is to find a good general lower bound. We achieve this for $N = 3$, and for the case $m = 0, N = 4$. These partial results then allow the construction of corresponding lower bounds based on reductions of the many-body problem respectively to scaled $N = 3$ and $N = 4$ systems. A formulation that unifies these results and all the known earlier partial results may be expressed as a lower-bound conjecture, which then becomes a theorem for each case that is proved.

2. Lower-bound conjecture

We first consider a model N-body Hamiltonian. This model has been constructed so that it essentially yields the corresponding nonrelativistic result in the limit $m \to \infty$. We are guided in the first instance by the centre-of-mass identity and inequality [7]

$$\sum_{i=1}^{N} \mathbf{p}_i^2 = \frac{1}{N} \sum_{1=i<j}^{N} (\mathbf{p}_i - \mathbf{p}_j)^2 + \frac{1}{N} \left(\sum_{i}^{N} \mathbf{p}_i \right)^2 \geqslant \frac{1}{N} \sum_{1=i<j}^{N} (\mathbf{p}_i - \mathbf{p}_j)^2. \qquad (2.1a)$$

For the corresponding semirelativistic problem, we lose this transparent algebraic inequality and must instead recover whatever can be proved to be true on the average. In a nutshell, this is the technical difficulty we must face in this paper. The Schrödinger N-body Hamiltonian H_S with the centre-of-mass kinetic energy removed and $\hbar = 1$ is therefore given by

$$H_S = \sum_{1=i<j}^{N} \left[\frac{1}{2mN} (\mathbf{p}_i - \mathbf{p}_j)^2 + V(r_{ij}) \right]. \qquad (2.2)$$

In Jacobi coordinates $[\rho] = B[\mathbf{r}]$, where B is an orthogonal $N \times N$ matrix with first row having entries all equal to $1/\sqrt{N}$, $\rho_2 = (\mathbf{r}_1 - \mathbf{r}_2)/\sqrt{2}$, and conjugate momenta $[\pi] = (B^t)^{-1}[\mathbf{p}] = B[\mathbf{p}]$, the equality in (2.1$a$) may be re-written simply as

$$\sum_{i=1}^{N} \mathbf{p}_i^2 = \pi_1^2 + \sum_{i=2}^{N} \pi_i^2. \qquad (2.1b)$$

Meanwhile, if $\Psi(\rho_2, \rho_3, \ldots, \rho_N)$ is a normalized translation-invariant N-boson wavefunction, we have [11, equations (6) and (7)]:

$$\left(\Psi, \pi_i^2 \Psi\right) = \left(\Psi, \pi_2^2 \Psi\right), \qquad \left(\Psi, \rho_i^2 \Psi\right) = \left(\Psi, \rho_2^2 \Psi\right), \qquad i = 2, 3, \ldots. \qquad (2.3)$$

We note parenthetically, for future reference, that with Jacobi coordinates we have the following explicit expression for $\mathbf{p}_N$:

$$\mathbf{p}_N = \frac{\pi_1}{\sqrt{N}} - \sqrt{\frac{N-1}{N}} \pi_N. \qquad (2.4)$$

When either the kinetic energy is a quadratic expression, as for all Schrödinger problems [7], or if the potential $V(r)$ is the harmonic oscillator $V(r) = kr^2$ [10], then these relations play a key role in the construction of a lower-bound model. Our purpose here is to make a reduction for the Salpeter problem and general $V(r)$, that is for problems for which neither the kinetic energy nor the potential energy has a simple quadratic form. We focus our attention on the kinetic energy since any progress made here would be potential independent. With these goals, the model N-body Hamiltonian we have constructed is given by

$$H_c = \sum_{1=i<j}^{N} \left[\sqrt{\gamma^{-1} (\mathbf{p}_i - \mathbf{p}_j)^2 + \left(\frac{2m}{N-1} \right)^2} + V(r_{ij}) \right] \qquad (2.5a)$$

or, equivalently,

$$H_c = \sum_{1=i<j}^{N} \left[\gamma^{-1} \sqrt{\gamma (\mathbf{p}_i - \mathbf{p}_j)^2 + (mN)^2} + V(r_{ij}) \right], \qquad (2.5b)$$

where $\gamma = \binom{N}{2} = \frac{1}{2}N(N-1)$ is the binomial coefficient. In the Schrödinger limit $m \to \infty$, we find $H_c \to mN + H_S$, where H_S is exactly the corresponding Schrödinger N-body Hamiltonian with the centre-of-mass kinetic energy removed, given in (2.2). Meanwhile, for the special case $N = 2$ of the semirelativistic problem itself we recover the well-known 2-body Salpeter Hamiltonian:

$$H = 2\sqrt{\left(\frac{\mathbf{p}_1 - \mathbf{p}_2}{2}\right)^2 + m^2} + V(r_{12}). \tag{2.6}$$

If we use new conjugate coordinates, we may write $r = \|\mathbf{r}\| = r_{12}$ and $p = \|\mathbf{p}\| = \|(\mathbf{p}_1 - \mathbf{p}_2)/2\|$, and then we have from (2.6)

$$H = 2\sqrt{p^2 + m^2} + V(r). \tag{2.7}$$

The idea is eventually to obtain an N-body lower bound which is the bottom of the spectrum of a scaled version of (2.7), namely

$$\mathcal{H} = \beta\sqrt{\lambda p^2 + m^2} + \gamma V(r), \qquad \beta, \lambda, \gamma > 0. \tag{2.8}$$

Meanwhile, the Salpeter Hamiltonian H itself is given by (1.1). We now suppose that Ψ is a translation-invariant normalized boson trial function. We consider expectations with respect to Ψ and we first observe that the permutation symmetry of Ψ implies the equality

$$\langle H_c \rangle = \langle \mathcal{H} \rangle, \qquad \text{where} \qquad \beta = N, \quad \lambda = \frac{2(N-1)}{N}, \quad \gamma = \frac{1}{2}N(N-1). \tag{2.9}$$

With these explicit values for the parameters $\{\beta, \lambda, \gamma\}$ in $\mathcal{H}$, we are now able to formulate the central idea of this paper explicitly, namely

Conjecture

$$\langle H \rangle \geqslant \langle \mathcal{H} \rangle. \tag{2.10}$$

This implies the following explicit conjectured lower energy bound

$$E \geqslant E_c^L = N \inf_{\psi}\left(\psi, \left[\left(\frac{2(N-1)}{N}p^2 + m^2\right)^{\frac{1}{2}} + \frac{N-1}{2}V(r)\right]\psi\right). \tag{2.11}$$

We can recover all earlier sharp bounds from this expression. We immediately recover the Schrödinger bounds [7] in the $m \to \infty$ limit (2.5). If we now assume that (2.11) is true as it stands for $m \geqslant 0$, and $V(r) = vr^2$, we recover our earlier bounds [9] for the semirelativistic oscillator. For $m > 0$, and $V(r) = -v/r$, we recover our earlier sharp bounds for the gravitational problem [11]. Meanwhile, the bounds we prove in the present paper establish a wider range of validity for this conjecture. For example, our theorem 3 establishes (2.11) for $m \geqslant 0$ and $N = 3$ in three dimensions; and theorem 4 establishes the case $m = 0$, $N = 4$. At present, we know of no counter example.

If we compare (2.5*b*) with (1.1) we see that the expectation of the difference may be written as

$$\langle H - H_c \rangle = \langle H - \mathcal{H} \rangle = \langle \delta(m, N) \rangle, \tag{2.12}$$

where

$$\delta(m, N) = \sum_{i=1}^{N}\sqrt{\mathbf{p}_i^2 + m^2} - \frac{2}{N-1}\sum_{1=i<j}^{N}\sqrt{\frac{N-1}{2N}(\mathbf{p}_i - \mathbf{p}_j)^2 + m^2}. \tag{2.13}$$

All our lower-bound results follow from the positivity (strictly speaking, non-negativity) of $\langle \delta(m, N \rangle$, when this can be established. We consider immediately the case $\{m = 0, N = 2\}$: the kind of reasoning we use in this case is generalized for the other cases. The approach we adopt is to think of the mean-value computation in momentum space where the momentum vectors $\mathbf{p}_i$ are multiplicative operators: these vectors form geometrical figures whose edges are the corresponding norms $\|\mathbf{p}_i\|$; mean values $\langle\|\mathbf{p}_i\|\rangle = d$ are considered last. For example, with $N = 2$, the three vectors $\{\mathbf{p}_1, \mathbf{p}_2, \mathbf{p}_1 - \mathbf{p}_2\}$ form the sides of a triangle. The observation that, as a consequence of the triangle inequality and boson symmetry, the largest possible value for $\langle\|\mathbf{p}_1 - \mathbf{p}_2\|\rangle$ is $2d$, then establishes positivity in this case. For $m > 0$ the argument must be adjusted accordingly. We shall consider this point in more detail in section 5, for the more interesting case $N = 3$ and $m > 0$. In order to prepare for what might be called 'stochastic geometry', we consider first $N = 3$ and $m = 0$, although this is a special case of the more general problem $m \geqslant 0$ discussed later. As we have remarked above, for the corresponding Schrödinger problem for general $V(r)$, or for the Salpeter harmonic-oscillator problem with $V(r) = kr^2$, a quadratic form is involved either in the kinetic- or the potential-energy term: for both of these problems, the conjecture follows as a result of the general quadratic mean-value identities (2.3) in Jacobi coordinates. For the Salpeter problems with general V, which is the subject of the present paper, the quadratic expressions (in momentum space) always appear inside the square-root sign, so these identities do not immediately apply. The general inequality $\langle\|\mathbf{p}\|\rangle \leqslant \langle\|\mathbf{p}\|^2\rangle^{\frac{1}{2}}$ does not remove this difficulty.

3. Proof in the case $m = 0, N = 3$

We have the following definition from (2.13):

$$\delta(0,3) = \|\mathbf{p}_1\| + \|\mathbf{p}_2\| + \|\mathbf{p}_3\| - \frac{1}{\sqrt{3}}\left(\|\mathbf{p}_1 - \mathbf{p}_2\| + \|\mathbf{p}_1 - \mathbf{p}_3\| + \|\mathbf{p}_2 - \mathbf{p}_3\|\right). \tag{3.1}$$

$$\langle\delta(0,3)\rangle = \left\langle\|\mathbf{p}_1\| + \|\mathbf{p}_2\| + \|\mathbf{p}_3\| - \frac{1}{\sqrt{3}}(\|\mathbf{p}_1 - \mathbf{p}_2\| + \|\mathbf{p}_1 - \mathbf{p}_3\| + \|\mathbf{p}_2 - \mathbf{p}_3\|)\right\rangle. \tag{3.2}$$

We note that $\delta(0,3)$ itself is negative for the choice $\mathbf{p}_2 = -\mathbf{p}_1 \neq \mathbf{0}$ and $\mathbf{p}_3 = \mathbf{0}$. However, this does not happen on the average. We have the following theorem.

Theorem 1. $\langle\delta(0,3)\rangle \geqslant 0$.

Proof. We know by boson symmetry that

$$\langle\|\mathbf{p}_1\|\rangle = \langle\|\mathbf{p}_2\|\rangle = \langle\|\mathbf{p}_3\|\rangle := k \tag{3.3}$$

and

$$\langle\|\mathbf{p}_1 - \mathbf{p}_2\|\rangle = \langle\|\mathbf{p}_1 - \mathbf{p}_3\|\rangle = \langle\|\mathbf{p}_2 - \mathbf{p}_2\|\rangle := q. \tag{3.4}$$

We may think of the $\{\mathbf{p}_i\}$, and their differences, as vectors, since they are used in momentum space where they become multiplicative operators. The six vectors in (3.1) are the six edges of a pyramid in $\Re^3$; the norms, $\|\mathbf{p}_i\|$ and $\|\mathbf{p}_i - \mathbf{p}_j\|$, are the corresponding lengths of these six pyramid edges. The permutation symmetry of the wavefunction implies the equalities (3.3) and (3.4). The mean difference $\langle\delta(0,3)\rangle$ is clearly smallest when the origin of the vectors $\{\mathbf{p}_i\}$ is at the centroid of the triangle formed by the differences $\{\mathbf{p}_i - \mathbf{p}_j\}$. In this minimal case we know from elementary geometry that $q = \sqrt{3}k$; consequently, $\langle\delta(0,3)\rangle = 0$. It follows that in general $\langle\delta(0,3)\rangle \geqslant 0$. This completes the proof for the case $m = 0$, $N = 3$. □

4. Proof for the case $m = 0, N = 4$

We consider the case $N = 4$ and $m = 0$ in (2.13). The six differences $\{\mathbf{p}_i - \mathbf{p}_j\}$ form a tetrahedron. The average lengths $q = \langle\|\mathbf{p}_i - \mathbf{p}_j\|\rangle$ are equal and force the tetrahedron to be regular. Meanwhile, the four mean lengths $k = \langle\|\mathbf{p}_i\|\rangle$ are again equal. This symmetry occurs when the **p**-origin is at the centroid of the tetrahedron, of, say, height h. For such a tetrahedron we have

$$h = \sqrt{\tfrac{2}{3}}q \qquad \text{and} \qquad k = \sqrt{\tfrac{3}{8}}q. \tag{4.1}$$

We may therefore write

$$\langle \delta(0,4) \rangle = 4\,\langle \|\mathbf{p}_1\| \rangle - 6\left(\tfrac{2}{3}\right)\sqrt{\tfrac{3}{8}}\,\langle \|\mathbf{p}_1 - \mathbf{p}_2\| \rangle = 4k - 4\sqrt{\tfrac{3}{8}}q = 0. \tag{4.2}$$

Thus we have the following.

Theorem 2. $\langle \delta(0,4) \rangle \geqslant 0$.

5. Proof in the case $m \geqslant 0, N = 3$

We consider

$$\begin{aligned}\delta(m,3) = {} & (\|\mathbf{p}_1\|^2 + m^2)^{\frac{1}{2}} + (\|\mathbf{p}_2\|^2 + m^2)^{\frac{1}{2}} + (\|\mathbf{p}_3\|^2 + m^2)^{\frac{1}{2}} - \left(\tfrac{1}{3}\|\mathbf{p}_1 - \mathbf{p}_2\|^2 + m^2\right)^{\frac{1}{2}} \\ & - \left(\tfrac{1}{3}\|\mathbf{p}_1 - \mathbf{p}_3\|^2 + m^2\right)^{\frac{1}{2}} - \left(\tfrac{1}{3}\|\mathbf{p}_2 - \mathbf{p}_3\|^2 + m^2\right)^{\frac{1}{2}}\end{aligned} \tag{5.1}$$

and

$$\begin{aligned}\langle \delta(m,3) \rangle = {} & \Big\langle (\|\mathbf{p}_1\|^2 + m^2)^{\frac{1}{2}} + (\|\mathbf{p}_2\|^2 + m^2)^{\frac{1}{2}} + (\|\mathbf{p}_3\|^2 + m^2)^{\frac{1}{2}} - \left(\tfrac{1}{3}\|\mathbf{p}_1 - \mathbf{p}_2\|^2 + m^2\right)^{\frac{1}{2}} \\ & - \left(\tfrac{1}{3}\|\mathbf{p}_1 - \mathbf{p}_3\|^2 + m^2\right)^{\frac{1}{2}} - \left(\tfrac{1}{3}\|\mathbf{p}_2 - \mathbf{p}_3\|^2 + m^2\right)^{\frac{1}{2}}\Big\rangle.\end{aligned} \tag{5.2}$$

Theorem 3. $\langle \delta(m,3) \rangle \geqslant 0$.

Proof. The three vectors $\mathbf{p}_i$, $i = 1, 2, 3$, and their three differences $\mathbf{p}_i - \mathbf{p}_j$ form six edges of a pyramid in $\Re^3$; the norms, $\|\mathbf{p}_i\|$ and $\|\mathbf{p}_i - \mathbf{p}_j\|$, are the corresponding lengths of these six pyramid edges. We now denote by T the triangle formed by the three difference edges $\{\|\mathbf{p}_i - \mathbf{p}_j\|\}$. For convenience, we shall think of T as lying in a horizontal plane and denote by P the top vertex of the pyramid; without loss of generality, we shall speak of P being above T. We let C be the point in the plane of T vertically under P. We now pick the vertex of T which contains $\mathbf{p}_1$, and call this V_1. In the plane of T, we construct a line from V_1 that is perpendicular to CV_1 and of length m, ending in the point U_1. We then join U_1 to P and observe that $\widehat{PV_1U_1} = \pi/2$. Similar constructions are now made with the other two vertices V_2 and V_3 of T; the three line segments U_iV_i are chosen to 'flow' in the same way round the pyramid axis CP. In fact, a new pyramid is formed by the three lines PU_i. By permutation symmetry we have that $\langle |PU_i| \rangle = k$ and $\langle |CU_i| \rangle = q$ where $i = 1, 2, 3$, and moreover

$$\langle (\|\mathbf{p}_i\|^2 + m^2)^{\frac{1}{2}} \rangle := k, \qquad i = 1, 2, 3, \tag{5.3}$$

and

$$\Big\langle \left(\tfrac{1}{3}\|\mathbf{p}_i - \mathbf{p}_j\|^2 + m^2\right)^{\frac{1}{2}} \Big\rangle := q, \qquad i, j = 1, 2, 3, \qquad i \neq j. \tag{5.4}$$

Since the position of P which minimizes k is C, and symmetry is obtained on the average, we conclude by elementary geometry that $k \geqslant q$. This inequality completes the proof of theorem 3. □

6. Application to $N \geqslant 3$

For $N \geqslant 3$, we can deduce a stronger lower bound than that provided by the $N/2$ bound, based on the result of section 5. If E and Ψ are the exact energy and corresponding wavefunction, we have $E = (\Psi, H\Psi)$ and therefore, by boson symmetry and theorem 3, we have

$$
\begin{aligned}
E &= \frac{N}{3}\left(\Psi, \left[\left(p_1^2+m^2\right)^{\frac{1}{2}} + \left(p_2^2+m^2\right)^{\frac{1}{2}} + \left(p_3^2+m^2\right)^{\frac{1}{2}}\right.\right. \\
&\quad \left.\left. + \frac{N-1}{2}\left(V(r_{12}) + V(r_{13}) + V(r_{23})\right)\right]\Psi\right) \\
&\geqslant N\left(\Psi, \left[\left(\frac{1}{3}p_{12}^2+m^2\right)^{\frac{1}{2}} + \frac{N-1}{2}V(r_{12})\right]\Psi\right) \\
&\geqslant N\left(\Psi, \left[\left(\frac{4}{3}p^2+m^2\right)^{\frac{1}{2}} + \frac{N-1}{2}V(r)\right]\Psi\right),
\end{aligned}
$$

where $\mathbf{r} = \mathbf{r}_1 - \mathbf{r}_2$ and $\mathbf{p} = \frac{1}{2}(\mathbf{p}_1 - \mathbf{p}_2) = \mathbf{p}_{12}$. Thus we have, for $N \geqslant 3, m \geqslant 0$, and $\|\psi(r)\| = 1$:

Theorem 4

$$
E \geqslant E^L_{N/3} = N \inf_{\psi}\left(\psi, \left[\left(\frac{4}{3}p^2+m^2\right)^{\frac{1}{2}} + \frac{N-1}{2}V(r)\right]\psi\right). \tag{6.1}
$$

In a similar fashion, we can relate the N-body problem for $N \geqslant 4$ and $m = 0$ to a reduced 4-body problem based on theorem 2. Specifically, we have for $N \geqslant 4, m = 0$, and $\|\psi(r)\| = 1$:

Theorem 5

$$
E \geqslant E^L_{N/4} = N \inf_{\psi}\left(\psi, \left[\left(\frac{3}{2}\right)^{\frac{1}{2}}\|\mathbf{p}\| + \frac{N-1}{2}V(r)\right]\psi\right). \tag{6.2}
$$

Theorems 4 and 5 summarize the principal results of this paper.

7. The linear potential $V(r) = r$ with $m = 0$

The lower bounds we have found all presume that the bottom of the spectrum of a scaled 1-body problem can be found. For Salpeter Hamiltonians, this task itself may not be trivially easy, although more tractable than for the many-body problem. For the operator $H = \|\mathbf{p}\| + r$ in three dimensions, we have at our disposal the accurate value $e = 2.2322$, for example, from the work of Boukraa and Basdevant [12] (the linear potential has also been considered by Pirner and Wachs [13] in an application to quark systems). By elementary scaling arguments, we therefore have for the 1-body problem

$$
H = ap + br \rightarrow E(a,b) = (ab)^{\frac{1}{2}}E(1,1) = (ab)^{\frac{1}{2}}e, \quad a, b > 0, \quad e = 2.2322. \tag{7.1}
$$

This relation will generate all the energy lower bounds for N-body problems with this potential. We shall use the notation $E^L_{N/2}$, $E^L_{N/3}$ and $E^L_{N/4}$, for the lower bounds given by equations (1.3), (6.1) and (6.2), and E_c for the conjectured bound (2.11). The formula (7.1) then allows us to derive formulae for these energies. Explicitly we find

$$E^L_{N/2} = N\left(\frac{N-1}{2}\right)^{\frac{1}{2}} e,\ N \geqslant 2 \tag{7.2a}$$

$$E^L_{N/3} = N\left(\frac{N-1}{\sqrt{3}}\right)^{\frac{1}{2}} e,\ N \geqslant 3 \tag{7.2b}$$

$$E^L_{N/4} = N\left(\frac{3(N-1)^2}{8}\right)^{\frac{1}{4}} e,\ N \geqslant 4 \tag{7.2c}$$

$$E^L_{c} = N\left(\frac{(N-1)^3}{2N}\right)^{\frac{1}{4}} e,\ N \geqslant 2. \tag{7.2d}$$

In order to find an upper bound, we follow [10] and use a Gaussian wavefunction, which we write initially in the form

$$\Phi(\rho_2, \rho_3, \ldots, \rho_N) = C\exp\left(-\frac{1}{2}\sum_{i=2}^{N}\rho_i^2\right) = \prod_{i=2}^{N}\phi(\rho_i), \quad C = \left(\frac{2}{\pi^{\frac{1}{4}}}\right)^{N-1}, \tag{7.3}$$

where the constant C is chosen to ensure the normalization of each radial factor ϕ on $L^2([0,\infty), r^2\,\mathrm{d}r)$. The boson symmetry of the trial function allows us to write $E \leqslant E^U_g = (\Phi, H\Phi)$, where we have

$$E^U_g = (\Phi, [N\|\mathbf{p}_N\| + \gamma V(\|\mathbf{r}_1 - \mathbf{r}_2\|)]\,\Phi)\,. \tag{7.4}$$

The identity (2.4) and the lemma proved in [9] (which allows us to remove the operator term π_1) imply

$$E^U_g = \left(\Phi, N\sqrt{\frac{N-1}{N}}\|\pi_N\| + \gamma V(\sqrt{2}\rho_2)\Phi\right). \tag{7.5}$$

The permutation symmetry of the Gaussian function in the *relative* coordinates and the factoring property allow us to replace π_N by $\pi_2 \equiv \sqrt{2}\mathbf{p}$. We write the conjugate variable to $\mathbf{p}$ as $\mathbf{r} \equiv \sqrt{2}\rho_2$, so that $V(r) = r$, and the wavefunction becomes $\phi(r)$. By introducing an additional scale parameter $\sigma > 0$, we then find

$$E^U_g = N\left(\sqrt{\frac{2(N-1)}{N}}\frac{1}{\sigma}\langle p\rangle + \frac{N-1}{2}\sigma\langle r\rangle\right). \tag{7.6}$$

Since the Gaussian radial function $\phi(r)$ is form invariant under the three-dimensional Fourier transformation, we have the equality

$$\langle p\rangle = \langle r\rangle = \frac{2}{\sqrt{\pi}}.$$

Since the minimum of the form $\alpha/\sigma + \beta\sigma$ over the scale $\sigma > 0$ is $2(\alpha\beta)^{\frac{1}{2}}$, we arrive at the following explicit formula for the Gaussian upper bound:

$$E^U_g = 4N\left(\frac{(N-1)^3}{2N\pi^2}\right)^{\frac{1}{4}}, \qquad N \geqslant 2. \tag{7.7}$$

We can immediately test this formula for the case $N = 2$ to obtain $E^U_g = 3.19154$, which is to be compared with the accurate numerical value $E = \sqrt{2}e = 3.1568$. More generally, we

Table 1. Ratios of upper to lower energy bounds $R_X = E_g^U/E_X^L$, where $X = N/2, N/3, N/4$; the ratio for the conjectured lower bound is $R_c = E_g^U/E_c^L$.

	$N=2$	$N=3$	$N=4$	$N=5$	$N=6$	$N=10$	$N\to\infty$
$R_{N/2}$	1.011	1.086 39	1.118 86	1.137 06	1.148 72	1.171 04	1.202 29
$R_{N/3}$		1.011	1.041 21	1.058 15	1.069	1.089 77	1.118 86
$R_{N/4}$			1.011	1.027 45	1.037 99	1.058 15	1.086 39
R_c	1.011	1.011	1.011	1.011	1.011	1.011	1.011

exhibit in table 1 ratios $R_X = E_g^U\big/E_X^L$, where X is $N/2$, $N/3$, $N/4$ or, for the conjectured lower bound, $R_c = E_g^U\big/E_c^L$. The percentage error in the determination of the energy by the bounds is approximately $50(R-1)\%$. The monotonic behaviour of R with N follows from the 'distance' of N from the size of the sub-system whose lower bound is best possible; if the conjecture were true, the quality of the lower bound would be the same for all N.

8. Conclusion

If a system of N identical particles is bound together by attractive pair potentials, the Hamiltonian H has N kinetic-energy terms and $\gamma = \binom{N}{2}$ potential terms. If the kinetic energy of the centre-of-mass can be subtracted off, then the number of kinetic-energy terms is reduced by one, and we would expect to obtain an expression of the form $E = \langle H\rangle = \langle (N-1)K + \gamma V\rangle$. The N-body energy E is then bounded below by the lowest energy $\mathcal{E}$ of a 'reduced' 1-body operator of the form $\mathcal{H} = (N-1)K + \gamma V$; if the boson-symmetry requirement of the N-body wavefunction is not too stringent, then this lower bound is at the same time a good approximation. This story is realized exactly for the nonrelativistic problem [7]: for the special case of the harmonic oscillator, $\mathcal{E}$ yields the exact energy E of the many-body system. The reduction details depend on the quadratic form of the nonrelativistic many-body kinetic-energy operator and the identities (2.3) for quadratic expressions in Jacobi relative coordinates.

For the semirelativistic counterpart, one generally loses the quadratic form in H and, along with it, the immediate reduction. An alternative reduction to the $H_{N/2}$ Hamiltonian is always possible and is important theoretically, but the resultant lower energy bound is weak. A quadratic form is returned to the potential in H in the special case of the harmonic oscillator, and this yields [10] a very sharp bound on the energy, though not now the exact solution, except in the Schrödinger limit $m \to \infty$. For general pair potentials, we have constructed a new Hamiltonian H_c that is translation invariant, both in coordinate and momentum space, and which reduces to the usual 2-body Hamiltonian for $N = 2$, and generally to $Nm + H_S$ in the large-m limit, where H_S is the Schrödinger Hamiltonian with the centre-of-mass kinetic energy removed. A reduction $\langle H_c\rangle = \langle\mathcal{H}\rangle \geqslant \mathcal{E}$ of H_c to a 1-body Hamiltonian $\mathcal{H}$ immediately follows. This is useful for the study of the many-body Hamiltonian H whenever it can also be established that $\langle H\rangle \geqslant \langle H_c\rangle$. We conjecture that this is always true. In the present paper, we have proved the conjecture for $N = 3$, and for $N = 4$ if $m = 0$; it is also true for the harmonic oscillator, and generally in the large-m limit. For the case of static gravity $V(r) = -1/r$, the conjecture yields the identical result to the energy bound we have established by a completely different argument, valid specially for this potential [11].

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